Game Theory Notes - Preferences*

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1 Preference Relations

The condition for an actor to make a rational choice is that the actor must have preferences that are complete and transitive. This entails that preferences allow for the sorting of all possible outcomes and options. Completeness ensures that when comparing possible choices, it is always clear which alternative is preferred. Transitivity, on the other hand, ensures that preferences do not exhibit cycles. If one prefers outcome a to b and b to c, then it follows that a is also preferred to c, avoiding cyclic preferences.

The concepts of utilities and payoffs were introduced in the preceding chapters, representing preferences with simple numbers for ranking utilities. However, in the real world, preferences are often more intricate.

Preferences

Definition (Preference relation): For any $x, y \in X$,

- x is weakly preferred to y, denoted as xRy, if and only if x is liked at least as much as y, indicating either a preference for x over y or indifference.
- x is strictly preferred to y, denoted as xPy, if and only if xRy and not yRx (i.e., x is weakly preferred to y, but y is not weakly preferred to x).
- x is indifferent to y, denoted as xIy, if and only if xRy and yRx (i.e., x is weakly preferred to y, and y is weakly preferred to x).

These concepts are analogous to the familiar $\geq$, $>$, and $=$ when dealing with real numbers.

Note: Alternative notations such as "$\succeq$," "$\succ$," and "$\approx$" can be used instead of R, P, and I, respectively.

A preference relation is a binary relation, a subset of the cross product of X and X, where X represents a set (e.g., the set of fruits: $X = \{$apple, banana, pear, ...$\}$).

- X is the set of all choice objects.
- $X \times X =$ the set of all pairs (x, y), representing all possible preferences.

*Notes based on the Game Theory lecture slides

In this context, a weak preference relation $(x, y) \in R$ denotes the preference of x over y, with $R \subseteq X \times X$.

Example: $R = \{(1,2),(2,3),(1,3),\ldots\}$ is the binary relation $<$ over $\mathbb{N}$ (the set of all pairs where the first entry is smaller than the second entry).

Rational Choice

What is a rational choice? An actor makes a rational choice if she chooses an alternative x^* such that x^* is weakly preferred to any other alternative. There can be none, exactly one, or many such x^*. If there is no weakly preferred alternative or if there are many such alternatives, making a rational choice might be difficult. However, if there is just one, the expectation is for the person to choose that weakly preferred alternative.

When determining whether someone can make a rational choice and which one it will be, the focus is on the set of alternatives that are weakly preferred to any other alternative. This set is known as the maximal set.

Definition (Maximal set): For any preference relation R on a choice set X, the maximal set is the set of all alternatives x that are weakly preferred (at least as good as) to all other alternatives in X:

$$M(R, X) = \{x \in X : xRy \text{ for any } y \in X\}$$

To understand the characteristics of preference relations and determine the nature of the maximal set (whether it has no elements, one element, or many elements), several properties are considered.

A preference relation R on X is:

- **Complete**: For all $x, y \in X$ where $x \neq y$, either xRy or yRx or both. Completeness ensures the ability to compare two different options.
- **Reflexive**: For all $x \in X$, xRx (if x is weakly preferred to x), indicating that x is at least as good as itself.
- **Transitive**: For all $x, y, z \in X$, if xRy and yRz, then xRz (if a is preferred to b and b to c, then a is preferred to c).
- **Quasi-transitive**: For all $x, y, z \in X$, if xPy and yPz, then xPz (i.e., if x is strictly preferred to y and y is strictly preferred to z, then x is strictly preferred to z).
- **Acyclic**: For any set $\{x_1, x_2, x_3, \ldots, x_n\} \subseteq X$ and $x_i P x_{i+1}$ for any $i < n$, $x_1 R x_n$. (i.e., if for any pair of consecutive options (x_i and x_{i+1}) you strictly prefer x_i to x_{i+1}, and if this is the case for all i, then x_1 is weakly preferred to x_n).

Definition 1.1 (Weak Ordering). *A weak ordering is a preference relation that is complete, reflexive, and transitive.*

These are also the conditions for making a rational choice: A person with weakly ordered preferences can make a rational choice. If a person has cycling preferences, where they prefer the 1st over the 2nd, the 2nd over the 3rd, but also the 3rd over the 1st, it is difficult to determine the most preferred option and make a choice. This definition is crucial, as weakly ordered preferences are assumed and used in most cases. "Acyclic" is less strict than "transitive," making it a weaker assumption (any transitive relation is also acyclic).

We can state this as a theorem that guarantees the existence of a rational choice. A rational choice is always possible when there is a weak ordering, which guarantees an element's existence in the maximal set. This is the core statement of the following theorem.

Theorem 1.1 (Existence)**.** *If X is a finite set, and R is a weak ordering on X, then the maximal set $M(R, X)$ is not empty.*

This does not imply that there is just one element, but it has at least one. And this guarantees a rational choice.

Remark: We can even show that the maximal set is not empty $M(R, X) \neq \emptyset$ if and only if R is complete, reflexive, and acyclic.

How do we translate this into practice? Often in political science, I have or come up with a story of what is going on, and then the question is: can I find a way to study this given the instruments that we know? Can I come up with a representation of the alternatives people are confronted with and of the preferences they have over these alternatives to study this situation in a way that fits the story?

Example: Suppose that you like beer and that you equally like whiskey, but you prefer having both at the same time even more. One pint of beer and a glass of whiskey cost two Euros each. You have four Euros in your pocket, and you are willing to spend the money now (that is to say, there are no opportunity costs). Can you make a rational choice? The set of alternative courses of action (and outcomes) is given by $X = \{(0,0), (1,0), (0,1), (2,0), (0,2), (1,1)\}$, where (x, y) denotes x pints of beer and y glasses of whiskey.

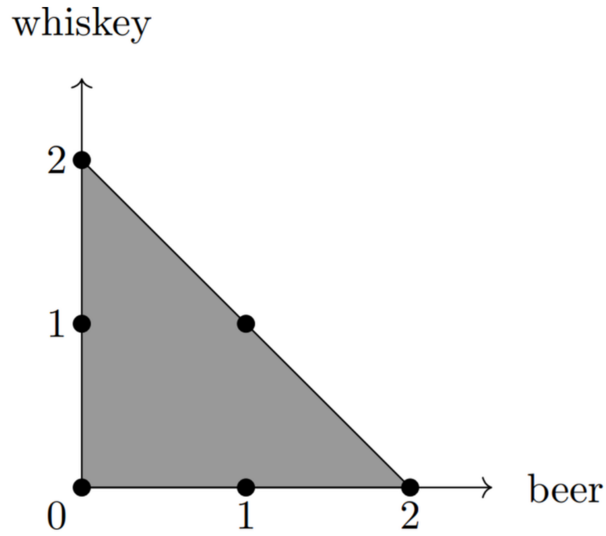
From the description above, we infer that the weak preference relation R consists of the following pairs

$$\begin{aligned}
&(1,0) \succeq (0,0), (2,0) \succeq (1,0), (2,0) \succeq (0,0),\\
&(0,1) \succeq (0,0), (0,2) \succeq (0,1), (0,2) \succeq (0,0),\\
&(1,1) \succeq (0,0), (1,1) \succeq (1,0), (1,1) \succeq (0,1),\\
&(1,1) \succeq (2,0), (1,1) \succeq (0,2),\\
&(0,0) \succeq (0,0), (1,0) \succeq (1,0), (2,0) \succeq (2,0),\\
&(0,1) \succeq (0,1), (0,2) \succeq (0,2), (1,1) \succeq (1,1)
\end{aligned}$$

(the final line ensures that the preference relation is reflexive; this information is not given but is reasonable, so we assume it. We often do not have all the information, so we need to make reasonable assumptions).

Then X is finite, and R is complete (reasonable that any pair can be compared), reflexive (by assumption), and transitive (because more is better than less). According to the theorem above, the maximal set $M(R, X)$ is not empty. It contains a single element: $M(R, X) = \{(1, 1)\}$.

Graphically, the dots denote the number of options to choose from, and preferences increase along the two dimensions (more is better than less) from the origins, but they also increase from $(0, 1)$ and $(1, 0)$ to the center at $(1, 1)$. So the point at the center is weakly preferred to the two other points.



The General Case: Continuous Choice Spaces

What can we say about the maximal set if there are infinite alternatives to choose from (e.g., the amount of money to spend on a political campaign, the income tax rate, etc.)? Then, the choice space is continuous and is any point in the previous triangle, where the line represents the budget constraint. With a continuous choice space, does it also hold that if preferences are weak orderings, there must be at least one element in the Maximal set?

Technically, we would identify these sets of alternatives with a subset of the real space $\mathbb{R}^n$; e.g., $X = [0, 1]$ for the tax rate, $X = [0, \infty)$ for campaign contributions, etc.

The conditions that guarantee the existence of a "best" alternative are much more demanding with a continuous choice space.

We get an idea of the difference between the finite and the continuous case when we consider the following problems:

- What is the maximal element from the natural numbers below 10? Okay, that's easy, it's 9.
- What is the maximal element from the real numbers below 10? Does not exist! Because the largest element would be 9.999... There is no way to say that I have a number that is the maximum in that set. The limit exists but is not in the set, so we have a limit but not a maximum.

It could be the case that (as in the example above) the choice set is in a way that the point $(1, 1)$ is not part of the choice set because, for example, you can spend any amount of money under 4 euros (up to 3.99...). That point could be at the limit but not included in the set. So the problem is when the ideal option we would like to pick is at the limit but not included in the choice set; and this is the problem with a continuous basis that does not exist with a finite basis, when all the choice options are in the set.

Theorem 1.2 (Conditions for the Existence of a Maximal Set)**.** *If there exists (1) a choice set that is part of* $\mathbb{R}^n$ *that is non-empty and is compact, and (2)* R *is not just a weak ordering (complete, reflexive, and transitive) but is also lower continuous, then the maximal set is not empty.*

Formally, if a subset $X \subseteq \mathbb{R}^n$ is non-empty and compact and R on X is complete, reflexive, transitive, and lower continuous, then $M(R, X) \neq \emptyset$.

What does it mean that the choice set is compact? It means that it is closed (e.g., $0 \leq x \leq 1$ is closed; 'all numbers below 10' is open) and bounded (if it does not go to infinity (e.g., $[0, 100)$ is bounded, $[0, \infty)$ is not)).

And **what does it mean for the preference relation to be (lower) continuous?** We use the definitions of upper and lower contour sets to define the continuity of the preference relation R. In political science, the upper contour set is often named *preferred to set.*

Definition 1.2. *Given a binary relation R on $\mathbb{R}^n$ the strict upper contour set of a point $x \in \mathbb{R}^n$ is $P(x) \equiv \{y \in \mathbb{R}^n : yPx\}$. The strict lower contour set of point x is the set $P^{-1}(x) \equiv \{y \in \mathbb{R}^n : xPy\}$. The level set of x is the set of points for which the agent is indifferent to x or $I(x) \equiv \{y \in \mathbb{R}^n : yRx \text{ and } xRy\}$.*

For any x, the upper contour set contains the points that are strictly preferred to x, the lower contour contains the points that x is preferred to, and the level set contains the points indifferent to x.

Definition 1.3. *A binary relation R on $\mathbb{R}^n$ is*

1. *upper continuous if $P(x)$ is open for all $x \in \mathbb{R}^n$*
2. *lower continuous if $P^{-1}(x)$ is open for all $x \in \mathbb{R}^n$*
3. *continuous if it is both lower and upper continuous*

In simpler terms, a lower continuous relation implies that if you have a sequence of points in $\mathbb{R}^n$ (the space of possible choices), and these points are converging to a particular point, the preference relation over these points should behave in a consistent way. In other words, the set of "better" choices should not suddenly change dramatically as you make small changes in the choices near x. The idea is that preferences do not "jump" or "flip" unexpectedly. If you have a sequence of points $x_1, x_2, \ldots$ converging to x, the points in the lower contour set of x will gradually "transition" as you approach x, without any sudden jumps or discontinuities.

We are trying to show here that we have additional conditions with a space that is not finite but continuous. Compactness and lower continuity ensure that the maximal set is not empty.

Theorem 1.3 (Conditions for the uniqueness of a maximal set). *If the subset X is not just convex and R is not just a weak ordering and upper continuous but also strictly convex, then the maximal set $M(R, X)$ is unique or is empty.*

Theorem 1.4 (Uniqueness). *If X is convex and R on X is strictly convex, then $M(R, X)$ is either empty or contains a single element.*

What does strictly convex mean? In a convex set, the line segment connecting any two points of the set lies entirely in the set. With strictly convex preferences, an actor who weakly prefers x to y strictly prefers any weighted average $(\lambda x + (1 - \lambda)y)$ to y.

Graphically, if we design it, the weighted average between x and y is the space between x and y. And this is preferred to y. So, any point that lies in the space between x and y is preferred to y. This is a concave function, i.e., a function that describes convex preferences.

2 Utility functions

The question arises: Why do we need utility functions when we already have preference relations? Preferences are fundamental, but working directly with them can be complicated. Utility functions simplify this process by assigning numerical values to outcomes or choice objects, facilitating comparisons through standard operators like " $\geq$ ".

A utility function, denoted as $u : X \to \mathbb{R}$, is sought such that:

- u(x) $\geq$ u(y) if and only if xRy (weak preference).
- u(x) $>$ u(y) if and only if xPy (strict preference).
- u(x) $=$ u(y) if and only if xIy (indifference).

Under what conditions does such a utility function exist? Are there universal rules governing its existence? The answer involves specific conditions.

Definition 1.4 (Utility function)**.** *Let X be a set of alternatives and R a weak preference ordering on X. We say that the utility function* u : X $\to \mathbb{R}$ *represents R if for any* $x, y \in$ X *:*

$$u(x) \geq u(y) \text{ if and only if } xRy$$

Notably, this single condition is sufficient because, in the earlier definition, strict preference P was defined in terms of weak preference R. If the single condition holds, the other two conditions hold as well.

Remark: For the maximal set, if u is a utility representation of R on X, then the maximal set is the set of all alternatives where the utility is maximized:

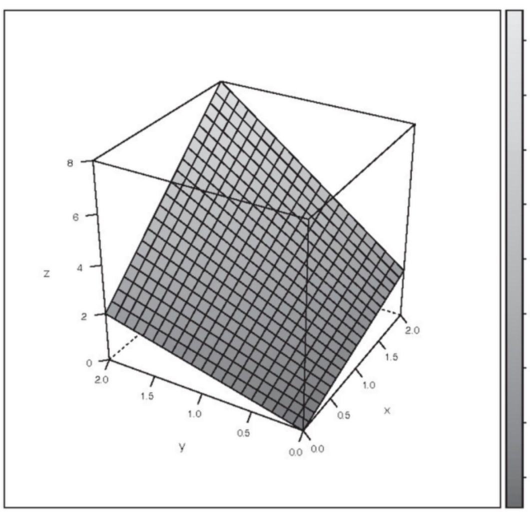
$$M(R, X) = \underset{x \in X}{\operatorname{argmax}}\ u(x)$$

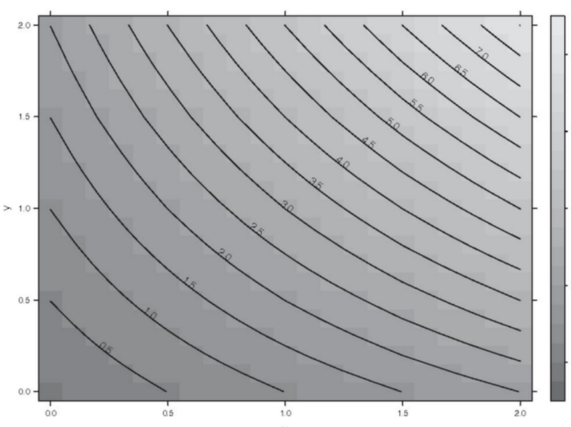
Returning to the beer-whiskey example, let's make it continuous. Assuming beer and whiskey are infinitely divisible goods, the set of alternatives to choose from is any point in the triangle:

$$X = \{(x, y) \mid x + y \leq 2\} \subseteq \mathbb{R}^2$$

This space is bounded (by budget constraints and does not go to infinity), closed (borders belong to the choice set), and convex (it is a triangle). An example representation of preferences R may be:

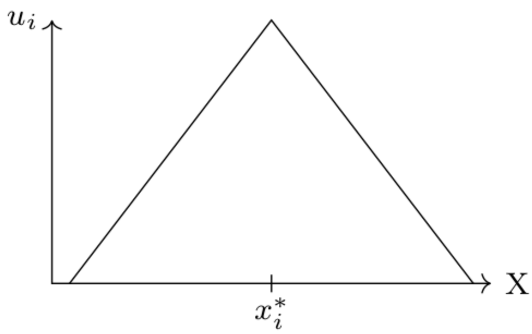
$$u : X \to \mathbb{R}, \quad u(x, y) = xy + x + y$$



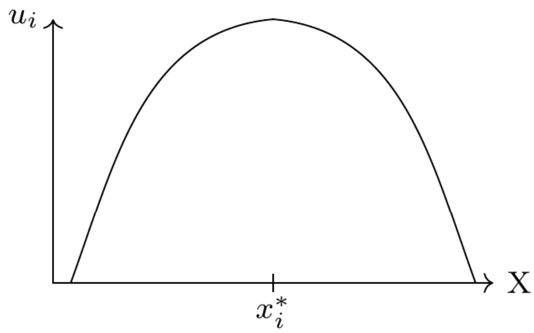


Indifference curves (figure to the right) are not linear because having both whiskey and beer simultaneously increases utility. The more, the better, but taking the two together is even better. The optimal point is where the utility is the largest in the entire space.

Single-peaked preferences are useful in many situations:



Linear utility function: $u_i(x) = -|x - x_i^*|$



Quadratic utility function: $u_i(x) = -(x - x_i^*)^2$

These functions represent the utilities of individuals over a specific policy. Usually, people have an ideal policy, and the more the policy deviates from this ideal point, the more the utility decreases.

In the example above, u_i are symmetric single-peaked utility functions, satisfying the following conditions:

- There is a single point where the utility to the left and right decreases. This is called the ideal point or position x_i^*.
- All points to the left and right with the same distance to x_i^* have the same utility.

Single-peakedness requires that we can order alternatives such that, for all individuals, the utility weakly decreases the further away from the ideal point.

Why Single-Peaked Preferences?

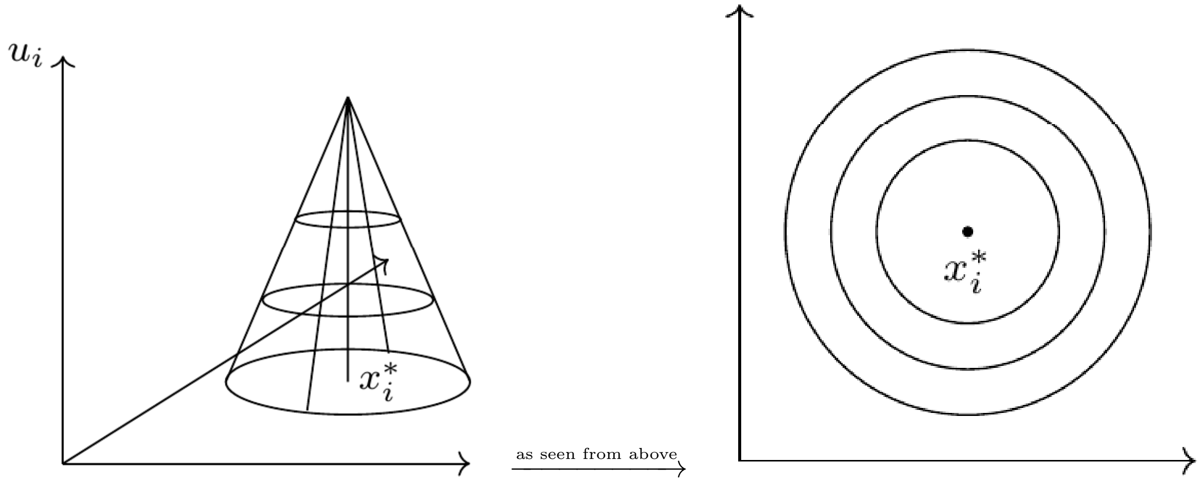
- Often plausible for preferences over public goods (policies), e.g., size of the defense budget, income tax rate, minimum wage.
- Easy to handle: preferences are (almost) uniquely characterized by the ideal point x_i^*.

However, often we have choice spaces that are slightly more complex than unidimensional ones. What do single-peaked preferences look like in a multidimensional space? In a unidimensional space, a policy is less preferred the more it is distant from the ideal point. The same holds true in a multidimensional space; the difference is that we need to consider the distance in a multidimensional space.

The commonly used distance in $\mathbb{R}^n$ is the Euclidean distance of $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$:

$$\|\mathbf{x} - \mathbf{y}\| = \sqrt{\sum_{j=1}^{n} (x_j - y_j)^2}$$

So, if we consider a linear utility function, e.g., $u_i(\mathbf{x}) = -\|\mathbf{x} - \mathbf{x}_i^*\|$, then it would look like this in :



(a) Linear single-peaked utility function in $\mathbb{R}^2$

(b) Indifference curves (curves connecting points between which i is indifferent)

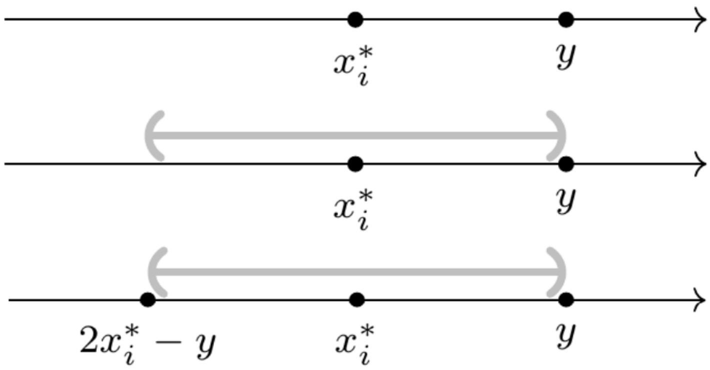
Utility decreases linearly with distance. In a three-dimensional figure where utility is the third dimension, this would be represented by a cone.

3 The Preferred-To Set

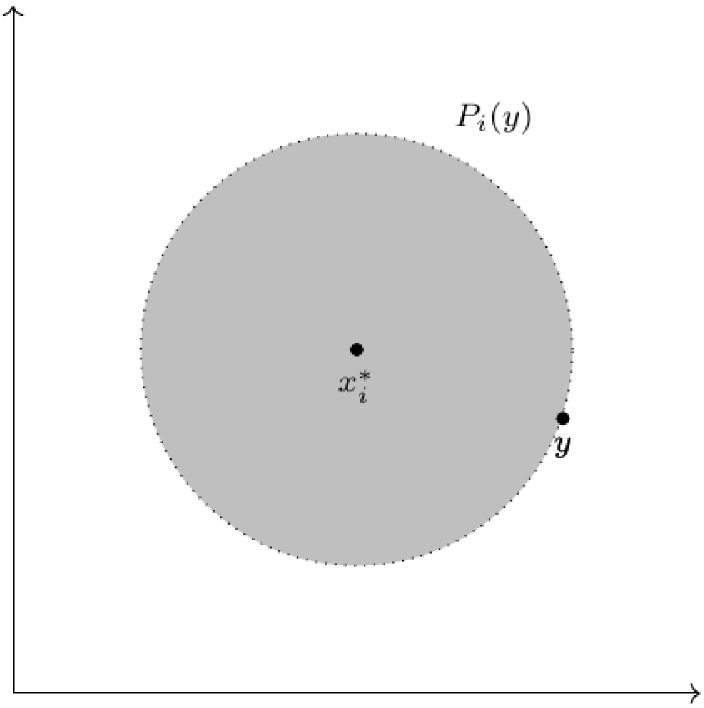
Definition 1.5. *Actor i 's preferred-to-set of y is the set of all $x \in X$ so that i strictly prefers x to y : $P_i(y) = \{x \mid xP_iy\}$.*

The preferred-to set of y is the set of all elements strictly preferred to y (all x strictly preferred to y). What are the alternatives that i strictly prefers to $y \in X$? In a unidimensional space in which x_i^*, the preferred-to set to y is:

- All points closer to x_i than y, excluding y.

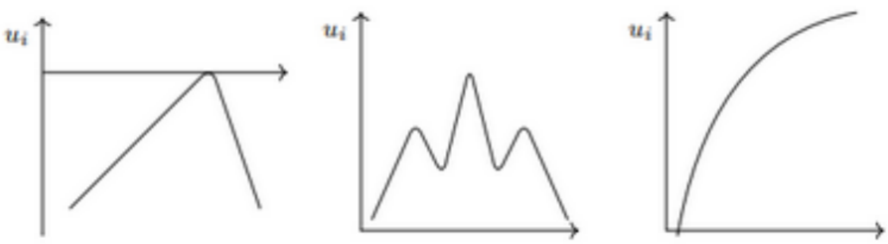


- In a two-dimensional space: that is all points in the interior of the circle (excluding the circle itself) through y : $\{x \mid \|\mathbf{x} - \mathbf{x}_i^*\| < \|\mathbf{y} - \mathbf{x}_i^*\|\}$.



- $P_i(y)$ is called i 's preferred-to-set of y. The definition is more general and applies to other preferences as well.

Other examples of utility functions:



1. The left one is single-peaked but not symmetric (utility decreases faster on the right-hand side).
2. The middle one is not single-peaked.
3. The right is single-peaked because there is a point with the highest utility at the right end of the choice space.

Summary

- **Modeling Preferences for Rational Choice**: Preferences form the foundation of decision-making models in game theory, as they represent the underlying structure of choices among alternatives.
- **Criteria for Rational Preferences**: For preferences to be rational, they must be complete, transitive, and sometimes acyclic, allowing for consistent choice-making and clear comparison among options.

- **Role of Maximal Sets and Utility Functions**: Maximal sets identify the best options under a preference relation, while utility functions quantify preferences, making complex choices analytically manageable.
- **Simplified Analysis with Single-Peaked and Single-Crossing Preferences**: These specific preference structures streamline the analysis of preferences, especially in one-dimensional or multidimensional policy spaces, by ensuring that preferences follow predictable patterns that avoid cyclical inconsistencies.